Bessel's equation:-

→ Bessel functions can be expressed in something like damped sines and cosines. This will be discussed in detail.

→ Applications in various range of topics - quantum mechanics wave motion, electrodynamics, heat, hydrodynamics, elasticity, etc.

Standard form of Bessel's equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y = 0 \quad \text{—— (1)}$$

$p \rightarrow$ constant called order of Bessel function y.

$y \rightarrow$ Solution of equation (1)

$p \rightarrow$ need not to be an integer

Eq. (1) using $x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$, can be written in more simpler form as

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2-p^2)y = 0 \quad \text{—— (2)}$$

We use method of series solution to obtain the soution of Bessel's equation. We assume solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+s} \quad \text{—— (3)}$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$x\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$$

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} \quad \text{--- (4)}$$

using (3), (4) in eqn. (2), ~~we obtain~~ and collect the Coefficients of the various powers of x. We write this expression in table form as-

	x^s	x^{s+1}	x^{s+2}	x^{s+3}	-- x^{s+n}
$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)$	s^2a_0	$(1+s)^2a_1$	$(2+s)^2a_2$	$(3+s)^2a_3$	-- $(n+s)^2a_n$
x^2y			a_0	a_1	--- a_{n-2}
$-p^2y$	$-p^2a_0$	$-p^2a_1$	$-p^2a_2$	$-p^2a_3$	--- $-p^2a_n$

--- (5)

Now the coefficient of x^s gives the indicial equation and value of s is obtained as

$$\left.\begin{aligned} s^2-p^2&=0, \\ s&=\pm p. \end{aligned}\right\} \quad \text{--- (6)}$$

Again the coefficient of x^{s+1} gives $a_1 = 0$ --- (7)

since $[(1+s)^2-p^2]a_1 = 0 \rightarrow a_1 = 0$

Next, from last column, we write (the coefficient of x^{n+s+n})

$$[(n+s)^2-p^2]a_n + a_{n-2} = 0$$

or $$a_n = -\frac{a_{n-2}}{(n+s)^2-p^2} \quad \text{—— (8)}$$

For $s=p$,

$$a_n = -\frac{a_{n-2}}{(n+p)^2-p^2} = -\frac{a_{n-2}}{n^2+2np}$$

or $$a_n = -\frac{a_{n-2}}{n(n+2p)} \quad \text{—— (9)}$$

From (7) $a_1 = 0 \Rightarrow$ only even term will survive

odd terms will be zero. (see eq. (8) or 9)

Therefore, we replace $n \to 2n$ and write eqn-(9) as

$$a_{2n} = -\frac{a_{2n-2}}{2n(2n+2p)}$$

or $$a_{2n} = -\frac{a_{2n-2}}{2^2 n(n+p)} \quad \text{—— (10)}$$

We use Γ- function to simplify above eq.

since $\Gamma(p+1) = p\,\Gamma(p)$ for any p.

$$\Gamma(p+2) = (p+1)\,\Gamma(p+1) = (p+1)\,p\,\Gamma(p)$$

$$\Gamma(p+3) = (p+2)\,\Gamma(p+2) = (p+2)(p+1)\,\Gamma(p+1)$$

From eq. (10) the coefficient can be written in terms of Γ function as

$$a_2 = \frac{a_0}{2^2(1+b)} = -a_0 \frac{\Gamma(1+b)}{2^2\Gamma(2+b)}$$

$$a_4 = -\frac{a_2}{2^3(2+b)} = \frac{a_0}{2^3(2+b)} \cdot \frac{\Gamma(1+b)}{2^2\Gamma(2+b)} = \frac{a_0\,\Gamma(1+b)}{2!\,2^4(2+b)(1+b)\Gamma(1+b)}$$

or $$a_4 = \frac{a_0}{2!\,2^4(1+b)(2+b)} = \frac{a_0\,\Gamma(1+b)}{2!\,2^4\,\Gamma(3+b)}$$

$$a_6 = -\frac{a_4}{3!\,2(3+b)} = -\frac{a_0\,\Gamma(1+b)}{3!\,2^6(3+b)\Gamma(3+b)}$$

$$= \frac{a_0\,\Gamma(1+b)}{3!\,2^6(3+b)(2+b)(1+b)\Gamma(1+b)}$$

$$= \frac{a_0}{3!\,2^6(1+b)(2+b)(3+b)}$$

or $$a_6 = \frac{-a_0\,\Gamma(1+b)}{3!\,2^6\,\Gamma(4+b)}$$

$\vdots$

Now using expression of coefficients $a_2, a_4, a_6 \ldots$, the series solution (for $s=b$) for Bessel's equal is given by

$$y = a_0 x^b + a_2 x^{2+b} + a_4 x^{4+b} + a_6 x^{6+b} + \ldots$$

or $y = x^{p}[a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots]$

or $y = x^{p}\left[a_0 - \frac{a_0 \Gamma(1+p)}{2^2 \Gamma(2+p)} x^2 + \frac{a_0 \Gamma(1+p)}{2!\, 2^4 \Gamma(3+p)} x^4 - \frac{a_0 \Gamma(1+p)}{3!\, 2^6 \Gamma(4+p)} x^6 + \cdots\right]$

or $y = x^{p} a_0 \Gamma(1+p)\left[\frac{1}{\Gamma(1+p)} - \frac{1}{\Gamma(2+p)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(3+p)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(4+p)}\left(\frac{x}{2}\right)^6 + \cdots\right]$

or $$\boxed{y = a_0 2^{p}\left(\frac{x}{2}\right)^{p}\Gamma(1+p)\left[\frac{1}{\Gamma(1)\Gamma(1+p)} - \frac{1}{\Gamma(2)\Gamma(2+p)}\left(\frac{x}{2}\right)^2 + \frac{1}{\Gamma(3)\Gamma(3+p)}\left(\frac{x}{2}\right)^4 - \frac{1}{\Gamma(4)\Gamma(4+p)}\left(\frac{x}{2}\right)^6 + \cdots\right]}$$

If we take $a_0 = \frac{1}{2^{p}\Gamma(1+p)} = \frac{1}{2^{p} p!}$

Then the solution y is called the Bessel function of first kind of order p and denoted by $J_p(x)$

$$J_p(x) = \frac{1}{\Gamma(1)\Gamma(1+p)}\left(\frac{x}{2}\right)^{p} - \frac{1}{\Gamma(2)\Gamma(2+p)}\left(\frac{x}{2}\right)^{2+p} + \frac{1}{\Gamma(3)\Gamma(3+p)}\left(\frac{x}{2}\right)^{4+p} - \frac{1}{\Gamma(4)\Gamma(4+p)}\left(\frac{x}{2}\right)^{6+p} + \cdots$$

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or

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+2)\,\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}}$$